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"Massive" Higher Spin Multiplets and Holography

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Abstract

We review the extrapolation of the single-particle string spectrum on $AdS_5 \times S^5$ to the Higher Spin enhancement point and the successful comparison of the resulting spectrum with the one of single-trace gauge-invariant operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We also describe how to decompose the common spectrum in terms of massless and massive representations of the relevant Higher Spin symmetry group.

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1   Introduction

We present an overview of the work done by one of the authors (M.B.) in collaboration with N. Beisert, J.F. Morales and H. Samtleben [1, 2, 3][1]. After giving some historical motivations for the interest in higher spin (HS) gauge fields and currents, we very briefly and schematically review some of the achievements of the holographic AdS/CFT correspondence[2]. We mostly but not exclusively focus on protected observables that do not change as we vary 't Hooft coupling constant $\lambda = g_{YM}^2 N$. We then discuss how the single-particle string spectrum on $AdS_5 \times S^5$ can be extrapolated to the HS enhancement point [9, 10, 11, 12, 13, 14] and how it can be successfully compared with the spectrum of single-trace gauge-invariant operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory [1, 2]. To achieve the goal we rely on the aid of Polya theory [15]. We also decompose the resulting spectrum in terms of massless and massive representations of the relevant HS symmetry group [3]. Eventually, we concentrate our attention on the generalization of the HS current multiplets, *i.e.* semishort multiplets, which saturate a unitary bound at the HS enhancement point and group into long ones as we turn on interactions. Finally, draw our conclusions and perspectives for future work. Properties of HS gauge theories are extensively covered by other contributions to this conference [16, 17, 18, 19, 20] as well as the reviews *e.g.* [21, 22, 23].

2   Historical motivations

The physical interest in HS currents dates back to the studies of QCD processes, such as deep inelastic scattering, where the structure of hadrons was probed by electrons or neutrinos. The process is studied at a scale $Q^2 = -q^2$, related to the momentum q transferred by the photon, which is much larger than the typical mass parameter of the theory Λ_{QCD}.

The fraction of momentum carried by the struck "parton", *i.e.* one of the hadron's constituents, is given by the Bjorken variable

$$0 \leq \xi = x_B = \frac{Q^2}{2P \cdot q} \leq 1 \,, \tag{1}$$

where P is the momentum of the hadron. Note that ξ is kinematically fixed. The optical theorem relates the amplitude of the process to the forward Comp-

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[1]A shorter account can be found in [4]

[2]For recent reviews see *e.g.* [5, 6, 7, 8].

ton amplitude $W^{\mu\nu}$

$$W^{\mu\nu} = W_1(x_B)\Big(\frac{q^\mu q^\nu}{q^2} - \eta^{\mu\nu}\Big) + W_2(x_B)\Big(P^\mu - q^\mu\frac{P\cdot q}{q^2}\Big)\Big(P^\nu - q^\nu\frac{P\cdot q}{q^2}\Big) =$$

$$= i\int d^4x\, e^{iqx}\int_0^1 \frac{d\xi}{\xi}\sum_i f_i(\xi)\langle q_i(\xi P)|T(J^\mu(x)J^\nu(0))|q_i(\xi P)\rangle\,, \qquad (2)$$

where $W_1(x)$ and $W_2(x)$ are scalar structure functions and $f_i(\xi)$ are the parton distribution functions, which depend on non-perturbative dynamical effects such as confinement. Parity-violating terms which can appear in weak interactions are omitted for simplicity. For non interacting spin $1/2$ partons, the structure functions satisfy Callan-Gross relations

$$\mathcal{I}mW_1 = \pi\sum_i e_i^2 f_i\,, \qquad\qquad \mathcal{I}mW_2 = \frac{4x_B}{Q^2}\mathcal{I}mW_1\,. \qquad (3)$$

Operator Product Expansion (OPE) yields

$$W^{\mu\nu} = \sum_i e_i^2\Big[\sum_{M=0}^\infty \frac{P^\mu P^\nu}{Q^2}\Big(\frac{2P\cdot q}{Q^2}\Big)^{M-2} A_i^{(n)}(Q^2) - \frac{1}{4}\eta^{\mu\nu}\sum_{M=0}^\infty\Big(\frac{2P\cdot q}{Q^2}\Big)^M A_i^{(n)}(Q^2)\Big]+\dots\,,$$
$$(4)$$

where the dominant contribution arises from operators with lowest twist $\tau = \Delta - s = 2+\dots$ The non-perturbative information is coded in the coefficients $A_i^{(n)}$ which can be related to the matrix elements in the hadronic state of totally symmetric and traceless HS currents built out of quark fields ψ

$$A_i^{(n)}: \qquad \langle P|\bar{\psi}_i\gamma^{(\mu_1}D^{\mu_2}\dots D^{\mu_n)}\psi_i|P\rangle \qquad (5)$$

Higher twist operators and flavour non-singlet operators, such as

$$\eta_{\lambda_i\lambda_j}\dots\eta_{\tau_\nu\rho_\tau}\dots\bar{\psi}_iD^{\mu_1}\dots D^{\mu_k}F^{\rho_1\nu_1}D^{\lambda_1}\dots D^{\lambda_\rho}F^{\rho_2\nu_2}\dots D^{\tau_1}\dots D^{\tau_\nu}\psi_j\,,$$

can appear in non-diagonal OPE's and produce mixing with purely gluonic operators. Their contribution to the OPE of two currents is suppressed at large Q^2.

2.1 Broken scale invariance

It is well known that QCD is only approximately scale invariant in the far UV regime of very large Q^2 where it exposes asymptotic freedom [24]. Scale

invariance is indeed broken by quantum effects, such as vacuum polarization that yields $\beta \neq 0$, and dimensional transmutation generates the QCD scale

$$\Lambda_{QCD} = \mu e^{-\frac{8\pi^2}{bg^2(\mu)}} \, . \tag{6}$$

The coefficient functions $A_i^{(n)}(Q^2)$ turn out to be Mellin transforms of the parton distributions and evolve with the scale Q^2 due to quantum effects. Depending on the parity of n, the parton and antiparton distributions contribute with the same or opposite sign. Defining $f_i^{\pm} = f_i \pm \bar{f}_i$, one has

$$A_i^{(n)}(Q^2) = \int_0^1 d\xi \xi^{n-1} f_i^+(\xi, Q^2) \tag{7}$$

in the case of even n and similarly

$$A_i^{(n)} = \int_0^1 d\xi \xi^{n-1} f_i^-(\xi, Q^2) \, , \tag{8}$$

in the case of odd n.

Parton distributions satisfy sum rules arising from global conservation laws stating *e.g.* that the net numbers of constituents of a given hadron do not depend on the scale

$$\int_0^1 d\xi f_i^-(\xi, Q^2) = n_i \, , \tag{9}$$

where n_i is independent of Q^2. For instance we know that protons are made of two up quarks $n_u = 2$ and one down quark $n_d = 1$ at each scale. Similarly the total momentum, including the gluonic contribution labelled by g, should be equal to the momentum of the hadron, and one obtains another sum rule of the form

$$\sum_i \langle x_i \rangle + \langle x_g \rangle = 1 \tag{10}$$

The evolution of the parton distributions is governed by Altarelli-Parisi (AP) equations [25][3]. For odd n, there is no operator mixing and the GLAP equations are "diagonal"

$$\frac{d}{dt} A_i^{(n)}(t) = \frac{\alpha_s(t)}{2\pi} \hat{P}^{(n)} A_i^{(n)}(t) \, , \quad t = \log Q^2 \quad , \tag{11}$$

To lowest order, the relevant kernel is

$$\hat{P}^{(n)} = \int_0^1 dz z^{n-1} P_{q\leftarrow q}(z) = \int_0^1 dz z^{n-1} \left(\frac{4}{3}\right) \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right] . \tag{12}$$

[3]Closely related equations were found by Gribov and Lipatov for QED [26].

that can be identified with the Mellin transform of the probability for a spin 1/2 parton to emit an almost collinear gluon. The result turns out to be simply given by

$$\hat{P}^{(n)} = -\frac{2}{3}\Big[1 + 4\sum_{k=2}^{n}\frac{1}{k} - \frac{2}{n(n+1)}\Big] \quad . \tag{13}$$

The presence of the harmonic numbers calls for a deeper, possibly number theoretic, interpretation and implies that the anomalous dimensions γ_S of HS currents at one loop behave as $\gamma_S \sim \log S$ for $S \gg 1$. Remarkably enough, the same leading behaviour holds true at two and higher loops [27].

For even n there is mixing with purely gluonic HS currents of the form

$$J_v^{(\mu_1\ldots\mu_n)|} = F_\lambda^{(\mu_1} D^{\mu_2} \ldots D^{\mu_{n-1}} F^{\mu_n)|\lambda} \quad , \tag{14}$$

In the holographic perspective twist two HS currents should correspond to nearly massless HS gauge fields in the bulk theory. Moreover, in $\mathcal{N} = 4$ SYM one has to take into account twist two currents that are made of scalars and derivatives thereof

$$J_\phi^{(\mu_1\ldots\mu_n)|} = \varphi_i D^{\mu_1} \ldots D^{\mu_n} \varphi^i \quad . \tag{15}$$

Although $\mathcal{N} = 4$ SYM theory is an exact superconformal field theory (SCFT) even at the quantum level, thanks to the absence of UV divergences that guarantees the vanishing of the β-function, composite operators can have non-vanishing anomalous dimensions.

2.2 Anomalous dimensions

We now turn to discuss anomalous dimensions and unitary bounds. In a CFT_D a spin S current with scaling dimension $\Delta = S + D - 2$ is necessarily conserved. For instance, a vector current with $S = 1$ and $\Delta = 3$ in $D = 4$ has a unique conformal invariant 2-point function of the form

$$\langle J^\mu(x) J^\nu(0)\rangle = (\partial^\mu\partial^\nu - \partial^2\delta^{\mu\nu})\frac{1}{x^4}, \qquad (D = 4) \quad , \tag{16}$$

that implies its conservation. For $\Delta = 3 + \gamma$ one finds instead

$$\langle \hat{J}^\mu(x) \hat{J}^\nu(0)\rangle = \frac{1}{x^{2\gamma}}(\partial^\mu\partial^\nu - \partial^2\delta^{\mu\nu})\frac{1}{x^4} \quad , \tag{17}$$

that leads to the (anomalous) violation of the current. Anomalous dimensions of HS currents satisfy positivity constraints. For instance, in $D = 4$, a scaling

operator carrying non-vanishing Lorentz spins j_L and j_R satisfies a unitary bound of the form

$$\Delta \geq 2 + j_L + j_R \tag{18}$$

At the threshold null states of the form $A = \partial^\mu J_\mu$ (dis)appear. When $j_L = 0$ or $j_R = 0$, *e.g.* for spin $1/2$ fermions and scalars, the unitary bound (18) takes a slightly different form

$$\Delta \geq 1 + j \quad . \tag{19}$$

The identity is the only (trivial) operator with vanishing scaling dimension.

In $\mathcal{N} = 4$ SYM the situation gets a little bit more involved [28, 29, 30, 31]. The highest weight state (HWS) of a unitary irreducible representation (UIR) of $(P)SU(2,2|4) \subset U(2,2|4) = SU(2,2|4) \times U(1)_B$ can be labelled by

$$\mathcal{D}\Big(\Delta, (j_L, j_R), [q_1, p, q_2]; B, C; L, P\Big), \tag{20}$$

where Δ is the dimension, (j_L, j_R) are the Lorentz spins, $[q_1, p, q_2]$ are the Dynkin labels of an $SU(4)$ R-symmetry representation. The central charge C, which commutes with all the remaining generators but can appear in the anticommutator of the supercharges, and the "bonus" $U(1)_B$ charge, related to an external automorphism of $SU(2,2|4)$, play a subtle role in the HS generalization of the superconformal group. The discrete quantum number P can be associated with the transposition of the gauge group generators or with a generalized world-sheet parity of the type IIB superstring. Finally the length L of an operator or a string state which is related to twist, but does not exactly coincide with it, is a good quantum number up to order one loop in λ.

Setting $C = 0$ and neglecting the $U(1)_B$ charge, there are three types of UIR representations of $PSU(2,2|4)$ relevant for the description of $\mathcal{N} = 4$ SYM theory.

- type A

  For generic (j_L, j_R) and $[q_1, p, q_2]$, one has

  $$\Delta \geq 2 + j_L + j_R + q_1 + p + q_2 \quad , \tag{21}$$

  that generalizes the unitary bound of the conformal group. The bound is saturated by the HWS's of "semishort" multiplets of several different kinds. Current-type multiplets correspond to $j_L = j_R = S/2$ and $p = q_1 = q_2 = 0$. Kaluza-Klein (KK) excitations of order p to $j_L = j_R = 0$ and $q_1 = q_2 = 0$. Above the bound, multiplets are long and comprise 2^{16} components times the dimension of the HWS.

- type B

  For $j_L j_R = 0$, say $j_L = 0$ and $j_R = j$, one has

  $$\Delta \geq 1 + j_R + \frac{1}{2}q_1 + p \quad , \tag{22}$$

  that generalizes the unitary bound of the conformal group. At threshold one finds 1/8 BPS multiplets.

- type C

  For $j_L = j_R = 0$ and $q_1 = q_2 = q$, one has

  $$\Delta = 2q + p \quad , \tag{23}$$

  the resulting UIR is 1/4 BPS if $q \neq 0$ and 1/2 BPS when $q = 0$. In the 1/2 BPS case, the number of components is $2^8 p^2(p^2-1)/12$, the multiplet is protected against quantum corrections and is ultrashort for $p \leq 3$. For $p = 1$ one has the singleton, with 8 bosonic and as many fermionic components, that corresponds to the elementary (abelian) vector multiplet. In the 1/4 BPS case, if the HWS remains a primary when interactions are turned on, the multiplet remains 1/4 BPS and short and protected against quantum corrections. For single trace operators, however, the HWS's all become superdescendants and acquire anomalous dimensions in a pantagruelic Higgs-like mechanism that deserves to be called *"La Grande Bouffe"*.

3 Lessons from AdS/CFT

Before entering the main part of the lecture, it may be useful to summarize what we have learned from the holographic AdS/CFT correspondence [5, 6, 7, 8]. Let us list some of the important lessons.

- The spectrum of 1/2 BPS single-trace gauge-invariant operators at large N matches perfectly well with the Kaluza-Klein spectrum of type IIB supergravity on S^5.

- The 3-point functions of chiral primary operators (CPO's) $\mathcal{Q}_p$, which are HWS's of 1/2 BPS multiplets,

$$\langle \mathcal{Q}_{p_1}(x_1)\mathcal{Q}_{p_2}(x_2)\mathcal{Q}_{p_3}(x_3)\rangle = C(p_1, p_2, p_3; N) \prod_{i<j} x_{ij}^{-2(l_i+l_j-\Sigma)} \ , \tag{24}$$

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are not renormalized by interactions and, as shown in [32], only depend on the quantum numbers p_i, associated with the spherical harmonics on S^5, and on the number of colors N, but neither on the gauge coupling g_{YM} nor on the vacuum angle ϑ_{YM} [33].

- There are some additional observables that are not renormalized. In particular, extremal and next-to extremal n-point correlators of CPO's are exactly the same as in the free theory [34]. A correlator of CPO's is (next-to) extremal when p_0 is the sum (minus two) of the remaining p_i and one finds

$$\langle Q_{l_0}(x_0)Q_{l_1}(x_1)\ldots Q_{l_n}(x_n)\rangle = G_{n+1}^{free}\,, \qquad l_0 = \sum_{i=1}^{n} l_i(-2)\,. \qquad (25)$$

- For near extremal correlators with $l_0 = \sum_i l_i(-4,-6,\ldots)$ one has partial non-renormalization [35]. In practise these correlation functions depend on lesser structures than naively expected in generic conformal field theory. Nevertheless these results are consequences of $PSU(2,2|4)$ invariance.

- Instanton effects $\mathcal{N}=4$ SYM correspond to D-instanton effects in type IIB superstring. In particular, certain higher derivative terms in the superstring effective action on $AdS_5 \times S^5$ are exactly reproduced by instanton dominated correlators on the boundary.

- (Partial) non-renormalization of "BPS" Wilson loops holds. For instance two parallel lines do not receive quantum corrections [36] while circular loops receive perturbative contributions only from rainbow diagrams [37] and non-perturbative contributions from instantons [38].

- The RG flows induced by deformations of the boundary CFT are holographically described by domain wall solutions in the bulk. In particular, it is possible to prove the holographic c-theorem. Indeed one can build a holographic c-function [39]

$$\beta^i = \dot{\phi}^i = \frac{\phi'(r)}{A'(r)}\,, \qquad \dot{c}_H = -G_{ij}\beta^i\beta^j \leq 0\,, \qquad (26)$$

and prove that it be monotonically decreasing along the flow.

- There is a nice way to reproduce anomaly in $\mathcal{N} = 4$ arising upon coupling the theory to external gravity or other backgrounds. In particular the holographic trace anomaly reads [40]

$$\langle \mathcal{T}_\mu^\mu \rangle_{(g_{\mu\nu})} = \frac{N^2}{4}(R_{\mu\nu}R^{\mu\nu} - \frac{1}{8}R^2) \,. \tag{27}$$

Quite remarkably, the structure of the anomaly implies

$$c_H = a_H \tag{28}$$

at least at large N. This is simple and powerful constraint on the CFT's that admit a holographic dual description. The techniques of holographic renormalization [41, 42, 43] have been developed to the point that one can reliably compute not only the spectrum of superglueball states [44] but also three-point amplitudes and the associated decay rates [45].

- Very encouraging results come from the recent work on string solitons with large spin or large charges [8], which qualitatively reproduce gauge theory expectations. In particular, the scaling

$$\sum_{k=1}^{S} \frac{1}{k} \sim \log S \tag{29}$$

for long strings with large spin S on $AdS_5 \times S^5$ has been found [8]. Moreover, the BMN limit [46], describing operators with large R charge, is believed to be dual to string theory on a pp-wave background emerging from the Penrose limit of $AdS_5 \times S^5$. For BMN operators, with a small number of impurities, light-cone quantization of the superstring suggests a close-form expression for the dimension as a function of coupling λ. For two-impurity BMN operators one has

$$\Delta = J + \sum_n N_n \sqrt{1 + \frac{\lambda n^2}{J^2}} \,. \tag{30}$$

where J is the R-charge.

4   Stringy $AdS_5 \times S^5$ and higher spin holography

In the boundary CFT the HS symmetry enhancement point is at $\lambda = 0$, so one may naively expect it to correspond to zero radius for $AdS_5 \times S^5$. Actually,

there might be corrections to $R^2 = \alpha'\sqrt{\lambda}$ for $\lambda \ll 1$ and we would argue that it is not unreasonable to expect the higher-spin enhancement point to coincide with the self-dual point $R^2 \sim \alpha'$. Ideally, one would like to determine the string spectrum by (covariant) quantization in $AdS_5 \times S^5$ background. However the presence of a R-R background has prevented a satisfactory resolution of the problem so far despite some progress in this direction [47]. Since we are far from a full understanding of stringy effects at small radius we have to devise an alternative strategy.

In [1] we computed the KK spectrum by naive dimensional reduction on the sphere and then extrapolated it to small radius, *i.e.* to the HS symmetry enhancement point. As we momentarily see, group theory techniques essentially determine all the quantum numbers, except for the scaling dimension, dual to the AdS mass. In order to produce a formula valid for all states at the HS point, we first exploit HS symmetry and derive a formula for the dimension of the massless HS fields. Then we take the BMN limit [46] as a hint and extend the formula so as to encompass the full spectrum. The final simple and effective formula does not only reproduce the HS massless multiplets as well as their KK excitations but does also describe genuinely massive states, which are always part of long multiplets. Finally, we compare the resulting string spectrum at the HS enhancement point in the bulk with the spectrum of free $\mathcal{N} = 4$ SYM theory on the AdS boundary. Clearly the matching of the spectrum is a sign that we are on the right track, but it is by no means a rigorous proof.

In order to study the string spectrum on $AdS_5 \times S^5$, we started with the GS formalism and built the spectrum of type IIB superstrings in the light-cone gauge in flat ten-dimensional space-time. The little group is $SO(8)$ for massless states and $SO(9)$ for massive ones. The chiral worldsheet supermultiplets are described by

$$Q_s = 8_v - 8_s\,, \qquad Q_c = 8_v - 8_c\,, \tag{31}$$

(see [1] for notations and details). At level 0 one has the massless supergravity multiplet

$$l = 0 \qquad Q_s Q_s = T_0 \qquad \text{(supergravity: } 128_B - 128_F) \tag{32}$$

At level 1 there are 2^{16} states as a result of the enhancement of $SO(8)$ to $SO(9)$ (128-fermions, 84-totally symmetric tensors and 44-antisymmetric)

$$l = 1 \qquad Q_s^2 Q_c^2 = T_1 \qquad (2^{16} \text{ states: } (44 + 84 - 128)^2) \tag{33}$$

Similarly at level l=2

$$l = 2 \qquad Q_c^2(Q_s + Q_s \cdot Q_s) = T_1 \times (1 + 8_v) = T_1 \times (9) \tag{34}$$

and so on

$$l \qquad \ldots = T_1 \times v_l^2 \qquad (v_1 = 1, v_2 = 9, \ldots). \qquad (35)$$

The important thing is that after building the spectrum one has to rewrite each level of the spectrum in terms of T_1, comprising 2^{16} states. Eventually, T_1 turns out to provide us with a representation of the superconformal group. What remains per each chirality will be called v_l. Combining with the opposite world-sheet chirality one gets v_l^2 as ground states.

In order to extend the analysis to $AdS_5 \times S^5$, *i.e.* perform the naive KK reduction, requires identifying which kinds of representations of the S^5 isometry group $SO(6)$ appear associated to a given representation of $SO(5)$. The latter arises from the decomposition $SO(9) \rightarrow SO(4) \times SO(5)$ for the massive states in flat space-time. Group theory yields the answer: only those representations of $SO(6)$ appear in the spectrum which contain the given representation of $SO(5)$ under $SO(6) \rightarrow SO(5)$.

Thus, after diagonalizing the wave equation for the bulk fields

$$\Phi(x, y) = \sum X_{AdS}(x) Y_{S^5}(y)$$

the spectrum of a string on $AdS_5 \times S^5$ assembles into representations of $SU(4) \approx SO(6)$, which are essentially given by spherical harmonics, with AdS mass

$$R^2 M_\Phi^2 = \Delta(\Delta - 4) - \Delta_{min}(\Delta_{min} - 4) \leftrightarrow C_2[SU(2,2|4)]. \qquad (36)$$

More explicitly, the wave equation can be written as

$$(\nabla^2_{AdS_5 \times S^5} - M_\Phi^2)\Phi_{\{\mu\}\{i\}} = 0, \qquad \{\mu\} \in R_{SO(4,1)}, \quad \{i\} \in R_{SO(5)} \qquad (37)$$

and one gets

$$\Phi_{\{\mu\}\{i\}} = \sum_{[kpq]} X^{[kpq]}_{\{\mu\}}(x) Y^{[kpq]}_{\{i\}}(y), \qquad (38)$$

where $[kpq] \in SO(6)$, the isometry group of S^5 and

$$\nabla^2_{S^5} Y^{[kpq]}_{\{i\}} = -\frac{1}{R^2}\Big(C_2[SO(6)] - C_2[SO(5)]\Big) Y^{[kpq]}_{\{i\}}. \qquad (39)$$

The KK tower build on the top of $SO(5)$ representation is given the following direct sum of $SO(6)$ representation

$$KK_{[mn]} = \sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{p=m-r}^{\infty} [r+s; p; r+n-s] + \sum_{r=0}^{m-1}\sum_{s=0}^{n-1}\sum_{p=m-r-1}^{\infty} [r+s+1; p; r+n-s],$$
$$(40)$$

where $[mn]$ are the two Dynkin labels of an $SO(5)$-representation.

The remaining $SO(4,2)$ quantum numbers, required to perform a lift to HWS representations $\mathcal{D}$ of $PSU(2,2|4) \supset SO(4,2) \times SO(6) \supset SO(4) \times SO(5)$, are the Lorentz spins j_L and j_R, and the scaling dimension Δ

$$D\{\Delta; (j_L, j_R); [k, p, q]\} = (1 + Q + \ldots + Q^{16})\Psi^{(\Delta, j_L, j_R)}_{[k,p,q]}. \tag{41}$$

For instance, at level $l = 1$, which corresponds to $\mathcal{D}\{2, (00); [000]\} \equiv \hat{T}_1^{(2)}$, the spectrum of KK excitations assembles into an infinite number of $SU(2,2|4)$ representations

$$H_1^{KK} = \sum_{M=0}^{\infty} [0n0]^M_{(0,0)}\hat{T}_1^{(2)} = \sum_{M=0}^{\infty} \mathcal{D}\{\Delta_0 = 2 + n, (00), [0n0]\}. \tag{42}$$

The HWS's in this formula have dimensions $\Delta_0 = 2 + n$, spin-0 and belong to $SU(4)$ representation $[0n0]$, which describes exactly the spherical harmonics. As we have already mentioned, so far the dimension Δ_0, at the HS point, is postulated so as to get the correct massless HS fields in the bulk.

For $l = 2$ the situation is slightly more involved, because $\mathbf{9}$ in (34) in v_l is neither a representation of the $SO(9,1)$ nor a representation of the $SO(4,2) \times SO(6)$. In this case, the correct way to proceed is to first decompose $\mathbf{9} \to \mathbf{10} - \mathbf{1}$ and then $\mathbf{10} \to \mathbf{6} + \mathbf{4}$

$$\underline{9} \to [010]^1_{(00)} + [000]^1_{(\frac{1}{2}, \frac{1}{2})} - [000]^2_{(00)} \sim \underline{10} - \underline{1}. \tag{43}$$

The corresponding KK-tower has a form

$$H_2^{KK} = \sum_{M=0}^{\infty} [0n0]^M_{00} \times \hat{T}_1^{(2)} \times \{[020]^2_{(00)} + [101]^2_{(00)} + [000]^2_{(00)} + 2[010]^2_{(\frac{1}{2}, \frac{1}{2})} +$$

$$+ [000]^2_{(11)} + [000]^2_{(10)} + [000]^2_{(01)} + [000]^2_{(00)} + [000]^4_{(00)} - 2[010]^3_{(00)} - 2[000]^3_{(\frac{1}{2}, \frac{1}{2})}\}. \tag{44}$$

It is worth stressing that negative multiplicities cause no problem as they cancel in infinite sum over n, precisely when the dimension is chosen properly.

The analysis of higher levels is analogous though slightly more involved. One has

$$H_l = \sum_0^{\infty} [0n0]^M_{00} \times \hat{T}_l^{(2)} \times (v_l^2), \tag{45}$$

with the decomposition

$$v_l^2 = [000]^\Delta_{(l-1, l-1)} + \ldots. \tag{46}$$

4.1 Exploiting HS symmetry

The superconformal group $PSU(2,2|4)$ admits a HS symmetry extension, called $HS(2,2|4)$ extension [48, 12, 13, 14]. In [12, 13, 14] Sezgin and Sundell have shown that the superstring states belonging to the first Regge trajectory on AdS can be put in one to one correspondence with the physical states in the master fields of Vasiliev's theory [21]. The HS fields which have maximum spin, *i.e.* $S_{max} = 2l + 2$ at level l, are dual to twist 2 currents, which are conserved at vanishing coupling. Including their KK excitations, one is lead to conjecture the following formula for their scaling dimensions

$$\Delta_0 = 2l + n \tag{47}$$

at the HS enhancement point. Now, at $\lambda \neq 0$, as we said in the introduction, *"La Grande Bouffe"* happens, since HS multiplets start to "eat" lower spin multiplets. For example, the short and "massless" $\mathcal{N} = 4$ Konishi multiplet combines with three more multiplets

$$K_{\text{long}} = K_{\text{short}} + K_{\frac{1}{8}} + K^*_{\frac{1}{8}} + K_{\frac{1}{4}} \tag{48}$$

and becomes long and massive. The classically conserved currents in the Konishi multiplet are violated by the supersymmetric Konishi anomaly

$$\bar{D}^A_{\dot{\alpha}} \bar{D}^{B\dot{\alpha}} K = g Tr(W^{AE}[W^{BF}, \bar{W}_{EF}]) + g^2 D^\alpha_E D_{\alpha F} Tr(W^{AE} W^{BF}), \tag{49}$$

where W^{AB} is the twisted chiral multiplet describing the $\mathcal{N} = 4$ singleton. In passing, the anomalous dimension of the Konishi multiplet is known up to three loops [49, 50, 51]

$$\gamma_K^{\text{1-loop}} = 3\frac{g^2 N}{4\pi^2}, \quad \gamma_k^{\text{2-loop}} = -3\frac{(g^2 N)^2}{(4\pi^2)^2}, \quad \gamma_k^{\text{3-loop}} = 21\frac{(g^2 N)^4}{(4\pi^2)^4} \quad, \tag{50}$$

whereas the anomalous dimensions of many other multiplets were computed by using both old-fashioned field-theoretical methods as well as modern and sophisticated techniques based on the integrability of the super-spin chain capturing the dynamics of the $\mathcal{N} = 4$ dilatation operator [51].

A systematic comparison with the operator spectrum of free $\mathcal{N} = 4$ SYM, may not forgo the knowledge of a mass formula encompassing all string states at the HS enhancement. Remarkably enough such a formula was "derived" in [2]. Consideration of the pp-wave limit of $AdS_5 \times S^5$ indeed suggests the following formula

$$\Delta = J + \nu, \tag{51}$$

where $\nu = \sum_n N_n$ and J is the R-charge emerging from $SO(10) \to SO(8)$ and N_n is the number of string excitations. Even though, the Penrose limit requires *inter alia* going to large radius so that the resulting BMN formula (51) is expected to be only valid for states with large R-charge J, (51) can be extrapolate to $\lambda \approx 0$ for all J's.

5 N=4 SYM spectrum: Polya(kov) Theory

In order to make a comparison of our previous results with the $\mathcal{N} = 4$ SYM spectrum we have to devise strategy to enumerate SYM states, and the correct way to proceed is to use Polya theory [15]. The idea was first applied by A. Polyakov in [11] to the counting of gauge invariant operators made out only of bosonic "letters".

Let us start by briefly reviewing the basics of Polya theory. Consider a set of words $A, B, \ldots$ made out of n letters chosen within the alphabet $\{a_i\}$ with $i = 1, \ldots p$. Let G be a group action defining the equivalence relation $A \sim B$ for $A = gB$ with $g \in G \subset S_n$. Elements $g \in S_n$ can be divided into conjugacy classes $[g] = (1)^{b_1} \ldots (n)^{b_n}$, according to the numbers $\{b_k(g)\}$ of cycles of length k. Polya theorem states that the set of inequivalent words are generated by the formula:

$$P_G(\{a_i\}) \equiv \frac{1}{|G|} \sum_{g \in G} \prod_{k=1}^{n} (a_1^k + a_2^k + \ldots + a_p^k)^{b_k(g)} . \tag{52}$$

In particular, for $G = Z_n$, the cyclic permutation subgroup of S_n, the elements $g \in G$ belong to one of the conjugacy classes $[g] = (d)^{\frac{n}{d}}$ for each divisor d of n. The number of elements in a given conjugacy class labelled by d is given by Euler's totient function $\varphi(d)$, that equals the number of integers relatively prime to and smaller than n. For $n = 1$ one defines $\varphi(1) = 1$. Computing P_G for $G = Z_n$ one finds:

$$P_n(\{a_i\}) \equiv \frac{1}{n} \sum_{d/n} \varphi(d)(a_1^d + a_2^d + \ldots + a_p^d)^{\frac{n}{d}} . \tag{53}$$

The number of inequivalent words can be read off from (52) by simply letting $a_i \to 1$.

For instance, the possible choices of "necklaces" with six "beads" of two different colors a and b, are given by

$$P_6(a, b) = \frac{1}{6}[(a + b)^6 + (a^2 + b^2)^3 + 2(a^3 + b^3)^3 + 2(a^6 + b^6)] =$$

$$= a^6 + a^5b + 3a^4b^2 + 4a^3b^3 + 3a^2b^4 + ab^5 + b^6\,,$$

and the number of different necklaces is $P_6(a = b = 1) = 14$.

We are now ready to implement this construction in $\mathcal{N} = 4$ theory, where the letters are the fundamental fields together with their derivatives

$$\partial^s \varphi^i; \partial^s \lambda^A; \partial^s \bar{\lambda}_A; \partial^s F^+; \partial^s F^-\,.$$

There are the **6** scalar fields φ^i, **4** Weyl gaugini λ^A, **4*** conjugate ones $\bar{\lambda}_A$ and the (anti) self-dual field strengths $F^\pm$. Since it is irrelevant for counting operators wether one is in free theory or not, we take as mass-shell conditions the free field equations. The single-letter on-shell partition functions then take the following form

$$Z_s(q) = \sum_\Delta n_\Delta^s q^\Delta = n_s \frac{q^{\Delta_s}}{(1-q)^4}(1-q^2)\,, \qquad \Delta_s = 1\,, \tag{54}$$

for the scalars such that $\partial^\mu \partial_\mu \varphi = 0$.

$$Z_f(q) = \sum_\Delta n_\Delta^f q^\Delta = n_f \frac{2q^{\Delta_f}}{(1-q)^4}(1-q)\,, \qquad \Delta_f = 3/2\,, \tag{55}$$

for the fermions with $\gamma^\mu \partial_\mu \lambda = 0$. For the vector field one gets a little bit involved expression, because apart from the equation of motion one has to take into account Bianchi identities $\partial^\mu F_{\mu\nu}^+ = \partial^\mu F_{\mu\nu}^- = 0$

$$Z_v(q) = \sum_\Delta n_\Delta^v q^\Delta = n_v \frac{2q^{\Delta_f}}{(1-q)^4}(3 - 4q + q^2)\,, \qquad \Delta_f = 2\,. \tag{56}$$

Taking statistics into account for $U(N)$ we obtain the free SYM partition function

$$Z_{YM} = \sum_{M=1}^N \sum_{d|n} \frac{1}{n} \varphi(d)[Z_s(q^d) + + Z_v(q^d) - Z_f(q^d) - Z_{\bar{f}}(q^d)]^{n/d}\,. \tag{57}$$

In fact, (57) is not exactly a partition function, rather it is what one may call the Witten index, wherein fermions enter with a minus sign and bosons with a plus sign. Now, words that consist of more than N constituents decompose into multi-trace operators, so representing $n = kd$, where d is a divisor of n and summing over k and d independently in the limit $N \to \infty$ one gets

$$Z_{YM} = - \sum_d \frac{\varphi(d)}{d} \log[1 - Z_s(q^d) - Z_v(q^d) + Z_f(q^d) + Z_{\bar{f}}(q^d)]\,. \tag{58}$$

For $SU(N)$, we have to subtract words that consist of a single constituent, which are not gauge invariant, thus the sum starts with $n = 2$ or equivalently

$$Z^{SU(N)} = Z^{U(N)} - Z^{U(1)}. \tag{59}$$

Finally, for $\mathcal{N} = 4$ we have $n_s = 6, n_f = n_{\bar{f}} = 4, n_v = 1$. Plugging into (57) and expanding in powers of q up to $\Delta = 4$ yields

$$Z_{\mathcal{N}=4}(q) = 21q^2 - 96q^{5/2} + 361q^3 - 1328q^{7/2} + 4601q^4 + \dots \tag{60}$$

5.1 Eratostene's (super)sieve

In order to simplify the comparison of the spectrum of SYM with the previously derived string spectrum, one can restrict the attention to superconformal primaries by means of Eratostenes's super-sieve, that allows us to get rid of the superdescendants. This procedure would be trivial if we knew that all multiplets were long, but unfortunately the partition function contains 1/2-BPS, 1/4-BPS, semishort ones too. The structure of these multiplets of $PSU(2,2|4)$ is more elaborated than the structure of long multiplets, which in turn is simply coded in and factorizes on the highest weight state.

Superconformal primaries, *i.e.* HWS of $SU(2,2|4)$, are defined by the condition

$$\hat{\delta}_S \mathcal{O} \equiv [\xi_A S^A + \bar{\xi}^A \bar{S}_A, \mathcal{O}] = 0, \tag{61}$$

where δ-is the supersymmetry transformation

$$\hat{\delta}_S = \delta_S - \delta_Q, \qquad (\eta = x - \xi)$$

$$\hat{\delta}_S \varphi^i = 0, \quad \hat{\delta}_S \lambda^A = \tau_i^{AB} \varphi^i \xi_B, \quad \hat{\delta}_S \bar{\lambda}_A = 0, \quad \hat{\delta}_S F_{\mu\nu} = \xi_A \sigma_{\mu\nu} \lambda^A$$

and τ_i^{AB} are 4×4 Weyl blocks of Dirac matrices in $d = 6$. The procedure can be implemented step by step using computer.

- Start with the lowest primaries – the Konishi scalar field $K_1 = tr\varphi_i\varphi^i$, and the lowest CPO $Q_{20'}^{ij} = tr\varphi^{(i}\varphi^{j)|})$

- Remove their superdescendants

- The first operator one finds at the lowest dimension is necessarily a superprimary.

- Go back to step 2

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We have been able to perform this procedure up to $\Delta = 11.5$ and the agreement with "naive" superstring spectrum

$$H_l = \sum_{n,l} [0n0]_{(00)} \times \hat{H}_l^{flat}$$

is perfect! Let us stress once more that our mass formula (51), though derived exploing HS symmetry and suggested by the BMN formula extrapolated to the HS enhancement point, reproduces semishort as well as genuinely long multiplets. The latter correspond to massive string states which never get close to being massless.

6 HS extension of (P)SU(2,2|4)

In the second part of this lecture, we identify the HS content of $\mathcal{N} = 4$ SYM at the HS enhancement point. Since we focus on the higher spin extension of superconformal algebra, it is convenient to realize $SU(2,2|4)$ by means of (super)oscillators $\zeta_\Lambda = (y_a, \theta_A)$ with

$$[y_a, \bar{y}^b] = \delta_a{}^b, \qquad \{\theta_A, \bar{\theta}^B\} = \delta_A{}^B, \tag{62}$$

where $y_a, \bar{y}^b$ are bosonic oscillators with $a, b = 1, \ldots, 4$ and $\theta_A, \bar{\theta}^B$ are fermionic oscillators with $A, B = 1, \ldots, 4$. The $su(2,2|4)$ superalgebra is spanned by various traceless bilinears of these oscillators. There are generators,

$$J_a{}^b = \bar{y}^a y_b - \frac{1}{2} K \delta^a{}_b, \qquad K = \frac{1}{2} \bar{y}^a y_a \tag{63}$$

which represents $so(4,2) \oplus u(1)_K$ subalgebra and generators

$$T^A{}_B = \bar{\theta}^A \theta_B - \frac{1}{2} B \delta^A{}_B, \qquad B = \frac{1}{2} \bar{\theta}^A \theta_A \tag{64}$$

which correspond $su(4) \oplus u(1)_B$. The abelian charge B is to be identified with the generator of Intriligator's "bonus symmetry" dual to the anomalous $U(1)_B$ chiral symmetry of type IIB in the AdS bulk. The Poincaré and superconformal supercharges are of the form

$$Q_a^A = y_a \theta^A, \qquad \bar{Q}_A^a = \bar{y}^a \theta_A. \tag{65}$$

The combination

$$C = K + B = \frac{1}{2} \bar{\zeta}^\Lambda \zeta_\Lambda. \tag{66}$$

is a central charge that commutes with all the other generators. Since all of the fundamental fields $\{A_\mu, \lambda_A^\alpha, \bar{\lambda}_{\dot\alpha}^A, \varphi^i\}$ have central charge equal to zero, we expect that local composites, just as well, have central charge equal to zero. So we consistently put

$$C = 0\,. \tag{67}$$

The higher spin extension $hs(2,2|4)$ is generated by the odd powers of the above generators

$$hs(2,2|4) = Env(su(2,2|4))/I_C = \bigoplus_{l=0}^{\infty} A_{2l+1}\,, \tag{68}$$

where I_C is the ideal generated by C and the elements J_{2l+1} in A_{2l+1} are of the form

$$J_{2l+1} = P_{\Sigma_1...\Sigma_{2l+1}}^{\Lambda_1...\Lambda_{2l+1}} \bar{\zeta}^{\Sigma_1}\ldots\bar{\zeta}^{\Sigma_{2l+1}}\zeta_{\Lambda_1}\ldots\zeta_{\Lambda_{2l+1}} - \text{traces}\,. \tag{69}$$

The singleton representation of $su(2,2|4)$ turns out to be also the singleton of $hs(2,2|4)$ in such a way that any state in the singleton representation of $hs(2,2|4)$ can be reached from the HWS by one step using a single higher spin generator. Note, that in $su(2,2|4)$ the situation is different, namely in order to reach a generic descendant from the HWS one has to apply several times different rasing operators.

6.1   sl(2) and its HS extension hs(1,1)

Since the $hs(2,2|4)$ algebra is rather complicated, in order to clarify the algebraic construction, we make a short detour in what may be called the $hs(1,1)$ algebra, the higher spin extension of $sl(2) \approx su(1,1)$.

Consider the $sl(2)$ subalgebra:

$$[J_-, J_+] = 2J_3\,, \qquad [J_3, J_\pm] = \pm J_\pm\,. \tag{70}$$

This algebra can be represented in terms of oscillators

$$J_+ = a^+ + a^+ a^+ a\,, \qquad J_3 = \frac{1}{2} + a^+ a\,, \qquad J_- = a\,, \tag{71}$$

where, as usual, $[a, a^+] = 1$ and the vacuum state $|0\rangle$ is annihilated by $J_- = a$. Other $sl(2)$ HWS's are defined by

$$J_- f(a^+)|0\rangle = 0 \quad \Rightarrow \quad f(a^+) = 1\,. \tag{72}$$

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Any state $(a^+)^n|0\rangle$ in this defining representation can be generated from its HWS $|0\rangle$ by acting with J_+^n. Therefore $f(a^+)$ defines a single irreducible representation of $sl(2)$, which will be called *singleton* and denoted by V_F. The $sl(2)$ spin of V_F is $-J_3|0\rangle = -\frac{1}{2}|0\rangle$. The dynamics of this subsector is governed by a Heisenberg spin chain.

The embedding of $sl(2)$ in $\mathcal{N} = 4$ SYM can be performed in different ways. In particular, the HWS can be identified with the scalar $Z = \varphi^5 + i\varphi^6$ and its $sl(2)$ descendants can be generated by the action of the derivative along a chosen complex direction, for instance $D = D_1 + iD_2$,

$$(a^+)^n|0\rangle \quad \leftrightarrow \quad D^n Z. \tag{73}$$

The tensor product of L singletons may be represented in the space of functions $f(a_{(1)}^+, \ldots, a_{(L)}^+)$. The resulting representation is no longer irreducible. This can be seen by looking for $sl(2)$ HWS's

$$J_- f(a_{(1)}^+, \ldots, a_{(L)}^+) = \sum_{s=1}^L \partial_s f(a_{(1)}^+, \ldots, a_{(L)}^+) = 0. \tag{74}$$

There is indeed more than one solution to these equations given by all possible functions of the form $f_L(a_{(s)}^+ - a_{(s')}^+)$. The basis for $sl(2)$ HWS's can be taken to be

$$|j_1, \ldots, j_{L-1}\rangle = (a_{(L)}^+ - a_{(1)}^+)^{j_1} \ldots (a_{(L)}^+ - a_{(L-1)}^+)^{j_{L-1}}|0\rangle, \tag{75}$$

with spin $J_3 = \frac{1}{2} + \sum_s j_s$. In particular for $L = 2$ one finds the known result

$$V_F \times V_F = \sum_{j=0}^\infty V_j, \tag{76}$$

where V_j is generated by acting with J_+ on the HWS $|j\rangle = (a_{(2)}^+ - a_{(1)}^+)^j|0\rangle$.

The higher spin algebra $hs(1,1)$ is generated by operators of the form

$$J_{p,q} = (a^+)^p a^q + \ldots. \tag{77}$$

The generators $J_{p,q}$ with $p < q$ are raising operators. In the tensor product of L singletons, HWS's of $hs(1,1)$ are the solutions of

$$\sum_{i=1}^L (a_{(i)}^+)^p \partial_i^q f(a_{(1)}^+, \ldots, a_{(L)}^+) = 0, \qquad p < q. \tag{78}$$

For $L = 2$ we can easily see that only two out of this infinite tower of HWS's survive for $j = 0$ and $j = 1$. That is all even objects belong to the same higher

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spin multiplet and all odd ones belong to another multiplet. For $L > 2$ one may consider either totally symmetric or totally antisymmetric representations. It can be easily shown that all of them are HWS's of HS multiplet.

$$\boxed{\ \ \ \ } \Rightarrow |0\rangle_{(L)} \sim Z^L, \quad \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} : \prod_{i<j}(a_i^+ - a_j^+)|0\rangle_{(L)} \sim (ZDZ\ldots D^{L-1}Z + a.s.)\,. \tag{79}$$

For more complicated Young tableaux, where L boxes distributed in different k columns it can be shown, that there is a solution of the form

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1}\end{array} = \prod_{p=1}^{k}\prod_{i_p<j_p}^{L_p}(a_{i_p}^+ - a_{j_p}^+)|0\rangle_{(L)} \leftrightarrow Z^{n_1}(DZ)^{n_2}\ldots(D^{n_s}Z) + \text{perms}\,. \tag{80}$$

The fact that (80) is indeed a solution is easily derived, however, its uniqueness is hard to prove.

The generalization to $hs(2,2|4)$ is almost straightforward for the totally symmetric representation

$$\boxed{\ \ \ \ } \quad \text{HWS:} \quad |0\rangle_{(L)} \leftrightarrow Z^L\,.$$

Namely, one starts with 1-impurity states

$$(WZ^{L-1} + \text{symm.}) = \frac{1}{L}J_{W\bar{Z}}^{HS}Z^L\,, \tag{81}$$

where the impurity (W) appears symmetrically in all places, and proceeds with 2-impurity states

$$(W_1 Z^{k-2} W_2 Z^{L-k} + \text{symm.}) = \frac{1}{L(L-1)}J_{W_1\bar{Z}}^{HS}J_{W_2\bar{Z}}^{HS}Z^L\,, \tag{82}$$

and so on. Note, that all operators of this symmetry are descendants of Z^L due to the fact, that each state in a singleton representation can be reached by a single step starting from the highest weight state.

For generic Young tableaux the task is more involved. However, the above construction goes through and the same arguments hold. For example, besides the descendants $J_{W\bar{Z}}$ of Z^L there are $L-1$ 1-impurity multiplets of states associated to the $L-1$ Young tableaux with $L-1$ boxes in the first row and

a single box in the second one. The vacuum state of HS multiplets associated to such tableaux can be taken to be $Y_{(k)} \equiv Z^k Y Z^{L-k-1} - Y Z^{L-1}$ with $k = 1, \ldots, L-1$. Any state with one impurity $Z^k W Z^{L-k-1} - W Z^{L-1}$ can be found by acting on $Y_{(k)}$ with the HS generators $J_{W\bar{Y}}$, where $J_{W\bar{Y}}$ is the HS generator that transforms Y into W and annihilates Z. The extension to other Young tableaux proceeds similarly though tediously.

6.2 HS content of $\mathcal{N} = 4$ SYM $\sim$ IIB at HS enhancement point

The free $\mathcal{N} = 4$ singleton partition function is given by the expression

$$Z_{\square} = \sum_{\Delta_s} (-1)^{2s} d_{\Delta_s} q^{\Delta_s} = \frac{2q(3 + \sqrt{q})}{(1 + \sqrt{q})^3} , \tag{83}$$

where Δ_s is the bare conformal dimension, *i.e.* the dimension at the HS enhancement point. Note that the singleton is not gauge invariant, thus one should build gauge invariant composites with two or more "letters". For example, the symmetric doubleton $\square\square$ can be obtained multiplying two singletons

$$\square \times \square = \square\square + \begin{array}{c}\square\\\square\end{array} , \tag{84}$$

where the antisymmetric diagram appears only in interactions and must be neglected in the free theory. The spectrum of single-trace operators in $\mathcal{N} = 4$ SYM theory with $SU(N)$ gauge group is given by all possible cyclic words built from letters chosen from $Z_{\square}$. It can be computed using Polya theory [15], which gives the generating function

$$Z(q, y_i) = \sum_{n>2} Z_n(q, y_i) = \sum_{n>2, d|n} u^n \frac{\varphi(d)}{n} Z_{\square}(q^d, y_i^d)^{\frac{n}{d}} \tag{85}$$

for cyclic words. The sum runs over all integers $n > 2$ and their divisors d, and $\varphi(d)$ is Euler's totient function, defined previously. The partition function (85) can be decomposed in representations of $hs(2, 2|4)$, *i.e.* HS multiplets. In particular, all operators consisting of two letters, assemble into the (symmetric) doubleton.

$$Z_2 \equiv Z_{\square\square}^{\delta_{a,b}} = \sum_n \chi(v_{2n}) . \tag{86}$$

For tri-pletons with three letters, one finds the totally symmetric tableau and the totally antisymmetric one.

$$Z_3 = Z_{\square\square\square}^{(d_{abc})} + Z_{\rlap{\square}\rlap{\square}\square}^{(f_{abc})} = \sum_{k=-\infty}^{+\infty} \sum_{n=0}^{\infty} c_n \Big(\chi(v_{2k}^n) + \chi(v_{2k+1}^{n+3}) + \chi(v_{2k+1}^n) + \chi(v_{2k}^{n+3}) \Big),$$

(87)

where the coefficients $c_n \equiv 1 + [n/6] - \delta_{n,1}$, yielding the multiplicities of $psu(2,2|4)$ multiplets inside $hs(2,2|4)$, count the number of ways one can distribute derivatives (HS descendants) among the boxes of the tableaux. Similarly for the tetra-pletons and penta-pletons one finds

$$Z_4 = Z_{\square\square\square\square}^{(q_{abcd})} + Z^{(d \cdot f)} + Z^{(f \cdot f)}$$

(88)

$$Z_5 = Z_{\square\square\square\square\square}^{(p_{abcde})} + Z_{\square\square\square} + 2Z_{\square\square\square} + Z_{\square\square} + Z$$

(89)

In the above partition functions, totally symmetric tableaux are to be associated to KK descendants of the HS doubleton multiplet. Other tableaux are those associated to lower spin Stückelberg multiplets, that are needed in order for *"La Grande Bouffe"* to take place. We checked that multiplicities, quantum numbers and naive dimensions are correct so that they can pair with massless multiplets and give long multiplets. Finally there are genuinely massive representations that decompose into long multiplets of $su(2,2|4)$ even at the HS point.

7 Conclusions and outlook

Let us summarize the results presented in the lecture.

- There is perfect agreement between the string spectrum on $AdS_5 \times S^5$ "extrapolated" to the HS enhancement point with the spectrum of single trace gauge invariant operator in free $\mathcal{N} = 4$ SYM at large N.

- The massless doubleton comprises the HS gauge fields which are dual to the classically conserved HS currents. Massive YT-pletons, *i.e.* multiplets associated to Young tableau compatible with gauge invariance, correspond to KK excitations, Stückelberg fields and genuinely long and massive HS multiplet. *"La Grande Bouffe"* is kinematically allowed to take place at $\lambda \neq 0$.

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- The one loop anomalous dimensions of the HS currents are given by

$$\gamma_S^{1-\mathrm{loop}} = \sum_{k=1}^{S} \frac{1}{k} \,,$$

and it looks likely that it have a number theoretical origin.

- There are some interesting issues of integrability. First of all the dilatation operator can be identified with the Hamiltonian of a superspin chain and is integrable at one loop or in some sectors up to two and three loops. Flat currents in $AdS_5 \times S^5$ give rise to a Yangian structure. Finally, the HS gauge theory can be formulated as a Cartan integrable system.

- There are some surprising features in $\mathcal{N} = 4$ SYM that have emerged from resolving the operator mixing at finite N [52]. In particular there are "unprotected" operators with $\gamma_S^{1-\mathrm{loop}} = 0$ and there are operators with nonvanishing anomalous dimension, whose large N expansion truncates at some finite order in N [53]

$$\gamma_s^{1\text{-loop}} = a + \frac{b}{N} + \frac{c}{N^2} \,,$$

with no higher order terms in $1/N$.

These and other facets of the AdS/CFT at small radius are worth further study in connection with integrability and HS symmetry enhancement. Sharpening the worldsheet description of the dynamics of type IIB superstrings on $AdS_5 \times S^5$ may turn to be crucial in all the above respects. Twelve dimensional aspects and two-time description [54] are worth exploring, too.

Acknowledgements

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